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Two-temperature elasto-thermo diffusive response inside a spherical shell with three-phase-lag effect

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ABSTRACT
The present work deals with the investigation of elasto-thermo diffusion interaction of a homogeneous isotropic spherical shell in the context of two-temperature generalized theory of thermo-elasticity with diffusion. The inner and outer boundaries of the spherical shell are free from stress and subjected to a time dependent thermal stoke. The chemical potential is also assumed to be a function of time on the boundary of the shell. The governing equations are solved in the Laplace transformation by using aoperator theory. The inverse of the transformed solution is carried out by applying a method of Bellman et al.. The stress, conductive temperature, displacement, mass concentration and chemical potential are computed numerically and presented graphically in a number of figures for copper material. A comparison of the results for three different models two-temperature Lord Shulman model (2TLS), two-temperature Green Naghdi model (2TGN-III) and two-temperature three-phase-lag model (2T3P) arealso presented for the different types of temperature field (one-temperature and two-temperature).
Keywords— Two-temperature generalized thermoelasticity, Three-phase-lag model, Mass diffusion, Chemical potential.

I. INTRODUCTION

The diffusion can be defined as the spontaneous migration of substances from regions of high concentration to regions of low concentration. The topic thermoelastic diffusion deals with the coupling effects of the fields of temperature, mass diffusion and strain, in addition to heat and mass exchange with the environment. It occurs as a result of the second law of thermodynamics which states that the entropy or disorder of any system must always increase with time. The recent interest in the study of this phenomenon is due to its extensive applications in geophysics and many industrial applications. The phenomenon of diffusion is used to improve the conditions of oil extractions (seeking ways of more efficiently recovering oil from oil deposits).

Biot [1] develops the coupled theory of thermo-elasticity to deal with defeat of the uncoupled theory that mechanical cause has no effect on the temperature field. In this theory, the heat equation has a parabolic form which predicts an infinite speed for the propagation of mechanical wave. The theory of generalized thermoelasticity with one relaxation time was introduced by Lord and Shulman [2]. This theory was extended by Dhaliwal and Sherief [3]. In the theory, the Maxwell-Cattaneo law of heat conduction replaces the conventional Fourier's law. For this theory, Ignaczak [4] studied the uniqueness of solution.

Thermo diffusion in the solids is one of the transport processes that have great practical importance. Most of the research associated with the presence of concentration and temperature gradients has been made with metals and alloys. Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, mass diffusion and strain fields. The first critical review was published in the work of Oriani (1969). Nowacki [5], [6], [7], [8] developed the theory of thermoelastic diffusion. In this theory, classical coupled thermoelastic model is used. Later on, Gawinecki et al. [9] and Gawinecki and Kacprzyk [10] proved a theory on uniqueness and regularity of the solution for a nonlinear parabolic thermoelastic diffusion problem. Sherief et al. [11] and, later on, Kumar and Kansal [12] introduced the generalized theories of thermoelastic diffusion in the frame of LS and GL theories by introducing thermal relaxation time parameters and diffusion relaxation time parameters into the governing equations, which allow the finite speeds of propagation of waves inside the medium. Sherief and Salah [13] investigated the problem of a thermoelastic half space in the context of the theory of generalized thermoelastic diffusion with one relaxation time. Aouadi [14], [15], [16], [17] also gave some attention on thermoelastic diffusion and generalized thermoelastic diffusion. The theory introduced by Sherief et al. [5], Kothari and Mukhopadhyay [18], presented the Galerkin-type representation of solutions for thermoelastic diffusion. In the context of the same theory, variational and reciprocity theorems have been established by Kumar et al. [19]. Different thermoelastic diffusion problems have been solved employing various models of generalized thermoelasticity by several researchers Tripathi et al. [20], Abbas et al. [21], Tianhu et al. [22].

The most recent and relevant development in thermo-elasticity theory is three-phase-lag model. Roychoudhuri [23] established this model, in which the Fourier's law of heat conduction is replaced by an approximation to a modified form with the introduction of different phase lags for the heat flux vector, temperature gradient and for the thermal displacement gradient. According to this model

$$\vec{q}(P,t+\tau_q) = -[K\vec{\nabla}T(P,t+\tau_T) + K^*\vec{\nabla}\nu(P,t+\tau_\nu)]$$

, where τ_q is the phase lag of heat flux, τ_T is the phase lag of temperature gradient, τ_ν is the delay time in thermal displacement gradient and K^* is the additional material constant. Three-phase-lag model is very useful in the problems of nuclear bonding, exothermic catalytic reactions, phonon-electron interactions, phonon-scattering etc.

The linearized version of the two-temperature theory (2TT) has been studied by many authors. Chen and Gurtin [24] and Chen et al. [25, 26] have formulated a theory of heat conduction in deformable bodies, which depends on two distinct temperatures (a) the conductive temperature ϕ and (b) the thermodynamic temperature θ. Lesan [27] has established uniqueness and reciprocity theorems for 2TT. The existence, structural stability and spatial behavior of the solution in 2TT have been discussed by Quintanilla [28]. The key element that sets the two-temperature thermoelasticity (2TT) apart from the classical theory of thermoelasticity (CTE) is the material parameter $\chi \geq 0$ called the temperature discrepancy [24].

Specifically, if χ=0, then $\phi=\theta$ and the field equations of the 2TT reduce to those of CTE. Although interest in the 2TT has waned since the 1970s, the recent contributions of Quintanilla [29, 30] and Puri and Jordan [31] have signaled something of a reversal in this trend. Youssef [32] has developed theory of two-temperature generalized thermoelasticity based on LS model. El-Karamany et al. [33] have established uniqueness and reciprocal principles in two-temperature Green-Naghdi thermoeelasticity theories. Quintanilla [34, 35] has proposed a modification of the 2TT that is based on dual-phase-lag and three-dual-phase-lag heat conduction, respectively. The constitutive law for the heat flux vector under the 2T3P model [36] is

$$\vec{q}(P,t+\tau_q) = -[K\vec{\nabla}\phi(P,t+\tau_T) + K^*\vec{\nabla}\nu(P,t+\tau_\nu)], \dot{\nu} = \phi$$

which is just the previous equation with the conductive temperature ϕ taking the place of T.

Problems related to the generalized thermoelasticity involving two temperatures have been investigated by Mondal et al. [37], Pal et al. [38], Islam et al. [39]. The aim of present paper is to investigate the effect of phase lags on elasto-thermo diffusion interactions in an isotropic elastic homogeneous spherical shell in the context of both one-temperature and two-temperature consideringLS, GN-III and 3P models. The analytical expressions for the displacement component, thermoelastic stresses, conductive temperature, mass concentration and chemical potential are obtained in the physical domain whose boundaries are traction free, are subjected to a time dependent temperature and chemical potential. The Laplace transform technique is used to obtain the general solution. To get the solution in the physical domain, the inversion of the transformed solution is carried out by applying the method of Bellman. The numerical estimates of the physical quantities are depicted graphically for a copper like material. A complete and comprehensive analysis and comparison of results are pre-temperature and two-temperature considering, GN-III and 3P models.
the introduction of the paper should explain the nature of the problem, previous work, purpose, and the contribution of the paper. The contents of each section may be provided to understand easily about the paper.

II. Formulation Of The Problem

We consider an isotropic homogenous thermo-elastic spherical shell with inner radius a and outer radius b in an undisturbed state and initially uniform temperature T_0. We introduce spherical polar coordinate (r, θ, ϕ) with the origin at the center O of the cavity. Since we consider thermoelastic interactions with the spherical symmetric, so all the functions considered will be function of the radial distance r and the time t only. It follows that the displacement vector $\vec{u}$, thermodynamic temperature θ and conductive temperature ϕ have the following forms:

$$\vec{u} = \left(u(r,t),0,0\right),\ \theta = \theta(r,t),\ \phi = \phi(r,t). \quad (1)$$

In the context of two temperature generalized thermoelastic diffusion based on three-phase-lag theory, the equation of motion, the equation of heat conduction and the equation of mass diffusion in absence of body forces for a linearly isotropic generalized thermoelastic solid are, respectively,

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)\frac{\partial e}{\partial r} - \beta_1\frac{\partial \theta}{\partial r} - \beta_2\frac{\partial C}{\partial r}, \quad (2)$$

$$[K^*\nabla^2\phi + \tau_\nu^*\nabla^2\dot{\phi} + K\tau_T\nabla^2\ddot{\phi}] = (1+\tau_q\frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2})(\rho c_E\ddot{\theta} + T_0\beta_1\ddot{e} + cT_0\ddot{C}), \quad (3)$$

$$D\beta_2\nabla^2 e + Dc\nabla^2\theta + \dot{C} + \tau^0\ddot{C} = Dd\nabla^2 C. \tag{4}$$

Where θ is the thermodynamic temperature, ϕ is the conductive temperature, λ and μ are Lame's constants, ρ is the density, C is the mass concentration. β_1, β_2 are the material constants given by

$\beta_1 = (3\lambda + 2\mu)\alpha_t, \beta_2 = (3\lambda + 2\mu)\alpha_c$, in which α_t and α_c are respectively, the coefficient of linear thermal expansion and linear diffusion expansion. K is the coefficient of thermal conductivity, K^* is the additional material constant; D is the diffusion coefficient and c_E is the specific heat at constant strain; τ_0 is the thermal relaxation time and τ^0 is the diffusion relaxation time, τ_T is the phase-lag of temperature gradient, τ_q is the phase-lag of heat flux, also $\tau_v^* = K + \tau_v K^*$, where τ_v is the phase-lag of thermal displacement gradient. c and d are the measures of the thermo-diffusion effect and diffusive effect, respectively. ∇^2 is the Laplacian, given in our case by,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

In equation (3):

(i) When $\tau_T = 0, \tau_v = 0, \tau_q = 0$, $\tau_v^* = K$, then this theory reduces to 2TGN-III model with diffusion.

(ii)When $K^* = 0$, $\tau_T = \tau_v = 0, \tau_v^* = K, \tau_q = \tau_0 > 0$ and $\tau_q^2 = 0$, then this theory reduces to 2TLS model with diffusion.

The strain components are given by

$$e_{rr} = \frac{\partial u}{\partial r,},\ e_{\theta\theta} = e_{\phi\phi} = \frac{u}{r}, \tag{5}$$

and thus the cubical dilatation will be

$$e = \frac{\partial u}{\partial r} + 2\frac{u}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u), \tag{6}$$

The constitutive equations are given by

$$\sigma_{rr} = 2\mu\frac{\partial u}{\partial r} + \lambda e - \beta_1\theta - \beta_2 C, \tag{7}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = (2\mu\frac{u}{r} + \lambda e - \beta_1\theta - \beta_2 C), \tag{8}$$

$$P = -\beta_2 e + dC - c\theta, \tag{9}$$

Where P is the chemical potential per unit mass of the diffusive material in the elastic body, σ_{ij} are the components of the stress tensor.

The relation between the conductive temperature ϕ and the thermodynamic temperature θ is given by,

$$\phi - \theta = \chi\nabla^2\phi. \tag{10}$$

Where χ (>0) is the two temperature parameter.

For convenience, the following dimensionless quantities are used:

$$(u', r') = c_1\eta(u, r), (\theta', \phi') = \frac{\beta_1(\theta,\phi)}{(\lambda+2\mu)},$$

$$C' = \frac{\beta_2 C}{(\lambda+2\mu)}, P' = \frac{P}{\beta_2}, t' = c_1^2\eta t, \sigma'_{ij} = \frac{\sigma_{ij}}{(\lambda+2\mu)},$$

$$\left[\tau'^0, \tau'_0, \tau'_q, \tau'_v, \tau'_T\right] = c_1^2\eta\left[\tau^0, \tau_0, \tau_q, \tau_v, \tau_T\right],$$

where $c_1^2 = \dfrac{(\lambda+2\mu)}{\rho}$ and $\eta = \dfrac{\rho c_E}{K}$. Therefore, the governing equations are given by equations(2)-(4) and (7)-(10) can be expressed in the following forms (dropping the primes for convenience), as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial e}{\partial r} - \frac{\partial\theta}{\partial r} - \frac{\partial C}{\partial r}, \tag{11}$$

$$[a_0(1+\tau_v\frac{\partial}{\partial t}) + \frac{\partial}{\partial t}(1+\tau_T\frac{\partial}{\partial t})]\nabla^2\phi = (1+\tau_q\frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2})[\ddot{\theta} + \varepsilon\ddot{e} + \varepsilon\alpha_1\ddot{C}], \tag{12}$$

$$\phi - \theta = \omega\nabla^2\phi, \tag{13}$$

$$\nabla^2 e + \alpha_1\nabla^2\theta + \alpha_2(\dot{C} + \tau^0\ddot{C}) = \alpha_3\nabla^2 C, \tag{14}$$

$$\sigma_{rr} = e - \frac{4}{\beta^2}\frac{u}{r} - \theta - C, \tag{15}$$

$$\sigma_{\theta\theta} = (1-\frac{2}{\beta^2})e + \frac{2}{\beta^2}\frac{u}{r} - \theta - C, \tag{16}$$

$$P = -e + \alpha_3 C - \alpha_1\theta. \tag{17}$$

Where,

$$a_0 = \frac{K^*}{Kc_1^2\eta},\ \alpha_1 = \frac{c\rho c_1^2}{\beta_1\beta_2},\ \alpha_2 = \frac{\rho c_1^2}{\eta D\beta_2^2},$$

$$\alpha_3 = \frac{d\rho c_1^2}{\beta_2^2},\ \varepsilon = \frac{T_0\beta_1^2}{\rho^2 c_1^2 c_E},\ \beta^2 = \frac{\lambda+2\mu}{\mu},$$

$$\omega = \chi c_1^2\eta^2.$$

2.1 BOUNDARY CONDITIONS:

We assume that the medium is initially at rest and undisturbed. All the initial state functions are therefore assumed to be zero. Now, to consider the thermo-diffusive interactions in the medium we assume that the boundary of the shell $(r = a, b)$ are traction free and is subjected to a thermal stoke. The chemical potential is also assumed to be a known function of time at the boundaries of the shell.

The boundary conditions are given by

(i) $\sigma_{rr}\big|_{r=a,b} = 0,$ (18)

(ii) $\phi = \phi_1 H(t)$: on $r = a, t > 0,$

$= \phi_2 H(t)$: on $r = b, t > 0.$ (19)

(iii) $P = P_1 H(t)$: on $r = a, t > 0,$

$= P_2 H(t)$: On $r = b, t > 0.$ (20)

Where ϕ_1, ϕ_2, P_1 and P_2 are the constants and $H(t)$ is the Heaviside unit-step function.

2.2 Solution in the Laplace transform domain:

Applying the Laplace transform defined by the relation,

$$\overline{f}(r,s) = \int_0^\infty f(r,t)e^{-st}dt \quad , \quad \mathrm{Re}(s) > 0,$$

to equation (11) and using homogeneous initial conditions, we get

$$s^2\overline{u} = \frac{\partial \overline{e}}{\partial r} - \frac{\partial \overline{\theta}}{\partial r} - \frac{\partial \overline{C}}{\partial r}. \qquad (21)$$

Now applying Laplace transform on equation (12) and using (13), we get

$$\overline{\theta} = \frac{1}{1+\omega a_3}\overline{\phi} - \frac{\varepsilon\alpha_1\omega a_3}{1+\omega a_3}\overline{C} - \frac{\varepsilon\omega a_3}{1+\omega a_3}\overline{e}, \qquad (22)$$

Where $a_3 = \dfrac{s^2(1+\tau_q s + \frac{1}{2}\tau_q^2 s^2)}{[a_0 + (1+a_0\tau_v)s + \tau_T s^2]}.$

Applying divergence operator on (21), we get

$$(\nabla^2 - s^2)\overline{e} = \nabla^2\overline{C} + \nabla^2\overline{\theta}. \qquad (23)$$

Using equation (22), (13)-(14) and (23) becomes

$$\nabla^2\overline{\phi} = M_1\overline{\phi} + M_2\overline{C} + M_3\overline{e}, \qquad (24)$$

$$[(1+M_1\varepsilon\omega)\nabla^2\overline{e}] - s^2\overline{e} = \frac{M_1}{a_3}\nabla^2\overline{\phi} + (1-M_2\omega)\nabla^2\overline{C}, \qquad (25)$$

$$(1-M_2\omega)\nabla^2\overline{e} + \frac{\alpha_1 M_1}{a_3}\nabla^2\overline{\phi} = [(\alpha_3 + \omega\alpha_1 M_2)\nabla^2 - \alpha_2 M_4]\overline{C}, \qquad (26)$$

where,

$$M_1 = \frac{a_3}{1+a_3\omega},\ M_2 = \frac{a_3\alpha_1\varepsilon}{1+a_3\omega},\ M_3 = \frac{a_3\varepsilon}{1+a_3\omega},$$

$$M_4 = s(1+\tau^0 s)$$

$$\overline{\sigma}_{rr} = \overline{e} - \frac{4}{\beta^2}\frac{\overline{u}}{r} - \overline{\theta} - \overline{C}, \qquad (27)$$

$$\overline{\sigma}_{\theta\theta} = (1-\frac{2}{\beta^2})\overline{e} + \frac{2}{\beta^2}\frac{\overline{u}}{r} - \overline{\theta} - \overline{C}, \qquad (28)$$

$$\overline{P} = -\overline{e} + \alpha_3\overline{C} - \alpha_1\overline{\theta}. \qquad (29)$$

Now, using equations (24)-(26), we obtain

$$\left(\nabla^6 - b_1\nabla^4 + b_2\nabla^2 - b_3\right)(\overline{e}, \overline{\phi}, \overline{C}) = 0. \qquad (30)$$

Where we used the notations

$$b_1 = \frac{1}{a_3^2(1-M_2\omega)^2 - a_3^2(\alpha_3 + M_2\alpha_1\omega)(1+M_1\varepsilon\omega)}[\alpha_1 M_1 a_3 M_3(1-M_2\omega) + \alpha_1 M_1 a_3 M_2(1+M_1\varepsilon\omega) + a_3^2\alpha_2 M_4 (1+M_1\varepsilon\omega) + M_1 a_3^2(1+M_1 s\omega)(\alpha_3 + M_2\alpha_1\omega) + a_3^2 s^2 (\alpha_3 + M_2\alpha_1\omega) + M_1 M_3 a_3(\alpha_3 + M_2\alpha_1\omega) + M_1 M_3 a_3 (1-M_2\omega) - M_1 a_3^2(1-M_2\omega)^2] \qquad (31)$$

$$b_2 = \frac{1}{a_3^2(1-M_2\omega)^2 - a_3^2(\alpha_3 + M_2\alpha_1\omega)(1+M_1\varepsilon\omega)}[\alpha_3 M_1 M_3 a_3 s^2 + M_1 a_3^2 s^2(\alpha_3 + M_2\alpha_1\omega) + a_3^2\alpha_2 M_4 M_1(1+ M_1\varepsilon\omega) + a_3^2 s^2\alpha_2 M_4 + \alpha_2 M_1 M_3 a_3 M_4] \qquad (32)$$

$$b_3 = \frac{1}{a_3^2(1-M_2\omega)^2 - a_3^2(\alpha_3 + M_2\alpha_1\omega)(1+M_1\varepsilon\omega)}[a_3^2\alpha_2 s^2 M_4 M_1] \qquad (33)$$

Equation (30) can be factorized as

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)(\overline{e}, \overline{\phi}, \overline{C}) = 0. \qquad (34)$$

Where k_1, k_2, k_3 are the roots with positive real part of the characteristic equation

$$k^6 - b_1 k^4 + b_2 k^2 - b_3 = 0,$$

and are given by [41]

$$k_1 = \sqrt{\frac{1}{3}[2p\sin q + b_1]}, \qquad (35)$$

$$k_2 = \sqrt{\frac{1}{3}[b_1 - p(\sqrt{3}\cos q + \sin q)]}, \qquad (36)$$

$$k_3 = \sqrt{\frac{1}{3}[b_1 + p(\sqrt{3}\cos q - \sin q)]}. \qquad (37)$$

Where

$$p = \sqrt{b_1^2 - 3b_2},\ q = \frac{\sin^{-1} v}{3},\ v = -\frac{2b_1^3 - 9b_1 b_2 + 27b_3}{2p^3}. \qquad (38)$$

Therefore, the solution of equation (30), which is bounded at infinity, is given by

$$\overline{\phi}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}[A_i(s)I_{1/2}(k_i r) + B_i(s)K_{1/2}(k_i r)], \qquad (39)$$

$$\overline{e}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}[A_i'(s)I_{1/2}(k_i r) + B_i'(s)K_{1/2}(k_i r)], \qquad (40)$$

$$\overline{C}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}[A_i''(s)I_{1/2}(k_i r) + B_i''(s)K_{1/2}(k_i r)],$$

(41)

where A_i, A_i', A_i'', B_i, B_i', B_i'' (for i =1, 2, 3) are parameters depending only on s. $I_{1/2}$ is the modified Bessel function of the first kind of order $1/2$ and $K_{1/2}$ is the modified Bessel function of second kind of order $1/2$. Now, from equations (24)-(26) and (39)-(41), we obtain the following relations:

$$A_i' = \frac{p_i}{d_i}A_i,\ A_i'' = \frac{f_i}{d_i}A_i,\ B_i' = \frac{p_i}{d_i}B_i,\ B_i'' = \frac{f_i}{d_i}B_i. \quad (42)$$

Where,

$$p_i = a_3(1-M_2\varepsilon\omega)k_i^4 + [M_1M_2 - a_3M_1(1-M_2\omega)]k_i^2,$$

$$d_i = k_i^2[a_3M_3(1-M_2\omega) + a_3M_2(1+\varepsilon M_1\omega)] - s^2a_3M_2,$$

$$f_i = a_3(1+\varepsilon M_1\omega)k_i^4 - [a_3M_1(1+\varepsilon M_1\omega) + s^2a_3 + M_1M_3]k_i^2 + s^2a_3M_1.$$

Thus we have

$$\bar{e}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}\frac{\varsigma_{1i}}{d_i}[A_i(s)I_{1/2}(k_ir) + B_i(s)K_{1/2}(k_ir)], \quad (43)$$

$$\bar{C}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}\frac{\varsigma_{2i}}{d_i}[A_i(s)I_{1/2}(k_ir) + B_i(s)K_{1/2}(k_ir)]. \quad (44)$$

Where

$$\varsigma_{1i} = a_3(1-M_2\varepsilon\omega)k_i^4 + [M_1M_2 - a_3M_1(1-M_2\omega)]k_i^2,$$

$$\varsigma_{2i} = a_3(1+\varepsilon M_1\omega)k_i^4 - [a_3M_1(1+\varepsilon M_1\omega) + s^2a_3 + M_1M_3]k_i^2 + s^2a_3M_1.$$

Using the relation between u and e from (6) and from (43), we get the solution for the dimensionless form of displacement as follows:

$$\bar{u}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}\frac{\varsigma_{1i}}{k_id_i}[A_i(s)I_{3/2}(k_ir) - B_i(s)K_{3/2}(k_ir)], \quad (45)$$

Therefore, from equations (27)-(29), (39) and (43)-(45), we get

$$\bar{\sigma}_{rr}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}A_i(s)[\varsigma_{3i}I_{1/2}(k_ir) - \frac{4p_i}{\beta^2rk_id_i} I_{3/2}(k_ir)] + \sum_{i=1}^{3}B_i(s)[\varsigma_{3i}K_{1/2}(k_ir) + \frac{4p_i}{\beta^2rk_id_i}K_{3/2}(k_ir)], \quad (46)$$

$$\bar{\sigma}_{\phi\phi}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}A_i(s)[\varsigma_{4i}I_{1/2}(k_ir) + \frac{2p_i}{\beta^2rk_id_i}I_{3/2}(k_ir)] + \sum_{i=1}^{3}B_i(s)[\varsigma_{4i}K_{1/2}(k_ir) - \frac{2p_i}{\beta^2rk_id_i}K_{3/2}(k_ir)], \quad (47)$$

$$\bar{P}(r,s) = \frac{1}{\sqrt{r}}\sum_{i=1}^{3}\varsigma_{5i}[A_i(s)I_{1/2}(k_ir) + B_i(s)K_{1/2}(k_ir)]. \quad (48)$$

Where, $\varsigma_{3i} = \frac{p_i}{d_i} - 1 - \frac{f_i}{d_i}$, $\varsigma_{4i} = (1-\frac{2}{\beta^2})\frac{p_i}{d_i} - 1 - \frac{f_i}{d_i}$,

$\varsigma_{5i} = -\frac{p_i}{d_i} - \alpha_1 - \frac{\alpha_3 f_i}{d_i}$.

To evaluate the unknown parameters, we shall use the Laplace transformation of the boundary condition (18)-(20), together with equations (39), (46) and (48); we have the following set of six linear equations in six unknowns:

$$\sum_{i=1}^{3}A_i(s)[\varsigma_{3i}I_{1/2}(k_ia) - \frac{4p_i}{\beta^2ak_id_i}I_{3/2}(k_ia)] + \sum_{i=1}^{3}B_i(s)[\varsigma_{3i}K_{1/2}(k_ia) + \frac{4p_i}{\beta^2ak_id_i}K_{3/2}(k_ia)] = 0, \quad (49)$$

$$\sum_{i=1}^{3}A_i(s)[\varsigma_{3i}I_{1/2}(k_ib) - \frac{4p_i}{\beta^2bk_id_i}I_{3/2}(k_ib)] + \sum_{i=1}^{3}B_i(s)[\varsigma_{3i}K_{1/2}(k_ib) + \frac{4p_i}{\beta^2bk_id_i}K_{3/2}(k_ib)] = 0, \quad (50)$$

$$\sum_{i=1}^{3}[A_i(s)I_{1/2}(k_ia) + B_i(s)K_{1/2}(k_ia)] = \frac{\phi_1\sqrt{a}}{s}, \quad (51)$$

$$\sum_{i=1}^{3}[A_i(s)I_{1/2}(k_ib) + B_i(s)K_{1/2}(k_ib)] = \frac{\phi_2\sqrt{b}}{s}, \quad (52)$$

$$\sum_{i=1}^{3}\varsigma_{5i}[A_i(s)I_{1/2}(k_ia) + B_i(s)K_{1/2}(k_ia)] = \frac{P_1\sqrt{a}}{s}, \quad (53)$$

$$\sum_{i=1}^{3}\varsigma_{5i}[A_i(s)I_{1/2}(k_ib) + B_i(s)K_{1/2}(k_ib)] = \frac{P_2\sqrt{b}}{s}. \quad (54)$$

We can obtain the $A_1(s)$, $A_2(s)$, $A_3(s)$, $B_1(s)$, $B_2(s)$, $B_3(s)$ by solving the above linear system of equations (49)-(54). This completes the solution of the present problem in the Laplace transform domain.

III. NUMERICAL RESULTS AND DISCUSSION:

In order to illustrate theoretical results in the preceding sections, we now present some numerical results. To get the solutions for the displacement, radial stress, shear stress, conductive temperature, thermodynamic temperature, chemical potential and mass concentration in the physical domain, we have to apply Laplace inversion formula to the equations (43)-(48) respectively. Here we adopt the method of Bellman et al. [40] for inversion and choose a time span given by seven values of time t_i, i =1 to 7 at which u_r, σ_{rr}, $\sigma_{\phi\phi}$, P and C are evaluated from the negative of logarithms of the roots of the shifted Legendre polynomial of degree 7. For the illustration we consider copper material with material constants. The physical data in SI units for which given as follows [41]:

$\lambda = 7.76\times10^{10}\,N.m^{-2},\ \mu = 3.86\times10^{10}\,N.m^{-2},$
$\rho = 8954 kg.m^{-3},\ K = 386 W.m^{-1}\mathrm{K}^{-1},$
$C_E = 383.1 J.kg^{-1}.\mathrm{K}^{-1},\ \mathrm{T}_0 = 293\mathrm{K},\ \alpha_1 = 5.43,$
$\alpha_c = 1.98\times10^{-4} m^3 kg^{-1},\ \alpha_t = 1.78\times10^{-5}\mathrm{K}^{-1},$
$\varepsilon = 0.0168,\ c = 1.2\times10^4 m^2\mathrm{K}^{-1}\mathrm{s}^{-2},$
$d = 0.9\times10^6 m^5 kg^{-1}\mathrm{s}^{-1},\ \alpha_2 = 0.533,\ \alpha_3 = 36.24,$
$\omega = 0.1.$

Also we have taken
$\phi_0 = 1, \tau_v = 0.1, \tau_q = 0.2, \tau^0 = 0.1, \tau_T = 0.15, \tau_0 = 0.01.$

and $R = 1$ for computational purposes. In G-N theory K^* is an additional material constant depending on the material. For copper material K^* is taken as $K^* = \dfrac{c_E(\lambda+2\mu)}{4}$.The computed results of the radial stress, shear stress, displacement, conductive temperature, mass concentration and chemical potential against r are displaced in figures 1-6, for 2T3P ($\omega = 0.1$) and 1T3P ($\omega = 0.0$); 2TGN-III ($\omega = 0.1$) and 1TGN-III ($\omega = 0.0$); 2TLS ($\omega = 0.1$) and 1TLS ($\omega = 0.0$) models respectively. The computations were carried out for times $t = 0.026$ and $t = 0.35$. The variation of the fields is observed when the step input of temperatures with φ_1=1 and φ_2=1 and step input of chemical potential with P_1=1 and P_2=1 are applied on the inner boundary $a = 1$ and outer boundary $b = 2$ of the shell. In all the graphs, the solid lines represent results for 3P model, the dotted lines represent results for GN-III model and the dashed lines represent results for LS model.

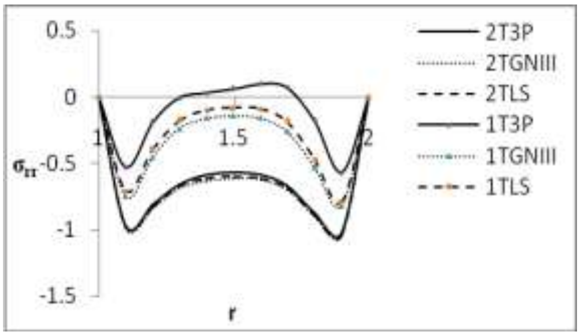

Fig. 1(a): Distribution of radial stress σ_{rr} at $t = 0.026$.

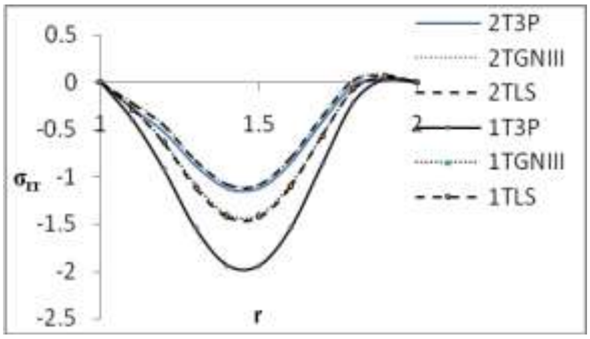

Fig. 1(b): Distribution of radial stress σ_{rr} at $t = 0.35$.

Figures 1(a) and 1(b) are plotted to show the variation of radial stress σ_{rr} against radial distance r in context of 3P, GN-III and LS models for one-temperature $(\omega = 0.0)$ and two-temperature $(\omega = 0.1)$ inside the spherical shell at times $t = 0.026$ and $t = 0.35$ respectively. At both the boundaries, the radial stress is noted to be zero, which agrees with the theoretical boundary conditions. At lower time $(t = 0.026)$ radial stress is fully compressive for two-temperature models and 1TGN-III and 1TLS models, whereas for 1T3P model it shows some positive values in the region $1.31 < r < 1.73$ but as time increases it becomes tensile near the boundaries and fully compressive at the middle zone in all cases. Moreover, the influence of diffusion is more significant near boundaries at lower time, whereas with the increase of time the region of influence shifts towards the middle of the shell under all consideration. The trend of variation of this field is almost similar for both one-temperature and two-temperature, but significantly higher magnitude of the radial stress is indicated for $\omega = 0.1$ as compared to $\omega = 0.0$ at lower time. Although σ_{rr} takes negative values in most of the considered region but this field shows some positive values near outer boundary at $t = 0.35$ for all the three models. The effect of diffusion is more prominent in case of 3P model in comparison with GN-III and LS models for both types of temperature field.

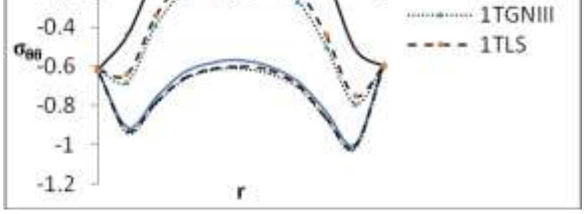

Fig. 2(a): Distribution of shear stress $\sigma_{\theta\theta}$ at $t = 0.026$.

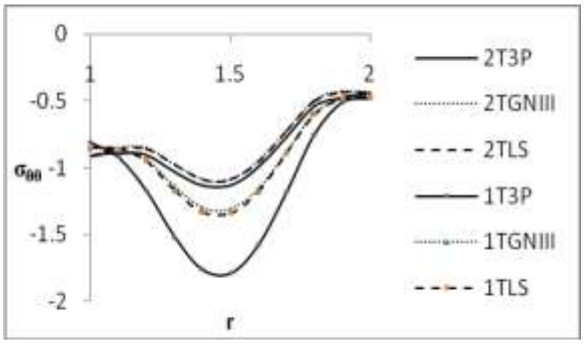

Fig. 2(b): Distribution of shear stress $\sigma_{\theta\theta}$ at $t = 0.35$.

Figures 2(a) and 2(b) represent the variation of shear stress $\sigma_{\theta\theta}$ against radial distance r inside the spherical shell for the same set of parameters considering 3P, GN-III and LS models. Shear stress is fully compressive for the current problem at all times. The magnitude of $\sigma_{\theta\theta}$ is large for two-temperature $(\omega = 0.1)$ when $t = 0.026$ and for one-temperature $(\omega = 0.0)$ when $t = 0.35$ for all three models, as demonstrated by radial stress. The effect of diffusion on this field increases significantly with increase of time under 3P, GN-III and LS models and this effect is visible at the boundaries, as well as in the middle zone of the shell. At lower time, the difference among the three models is noted to be prominent in the case of one-temperature than that of two-temperature. This difference decreases with the increase of time.

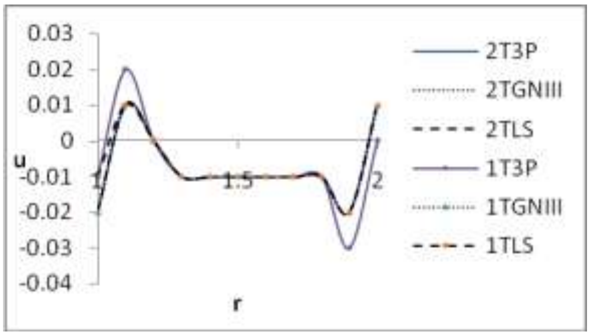

Fig. 3(a): Distribution of displacement u at $t = 0.026$.

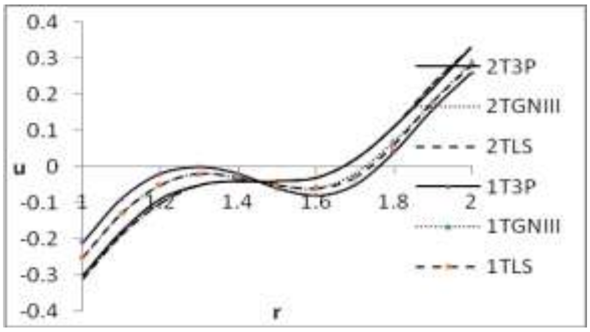

Fig. 3(b): Distribution of displacement u at $t = 0.35$.

In figures 3(a) and 3(b), the space variation of the displacement u inside the spherical shell are observed for 3P, GN-III and LS models in the case of both $\omega = 0.0$ and $\omega = 0.1$. The graphs of displacement under GN-III and LS models are almost merged together, but 3P model predicts a significantly different value as compared to the previous two models at both lower and higher time. A significant effect of diffusion on displacement field under 1T3P model is observed and this effect increases with the passage of time. The absolute value of u increases with the increase of time in all cases.

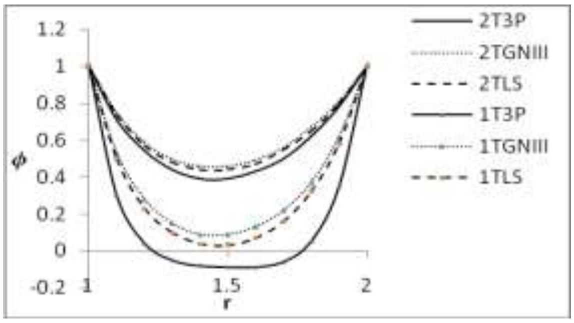

Fig. 4(a): Distribution of conductive temperature at $t = 0.026$.

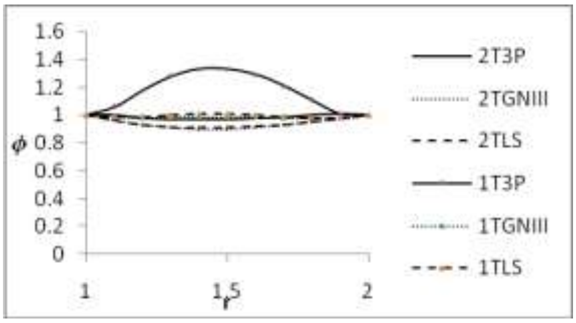

Fig. 4(b): Distribution of conductive temperature at $t = 0.35$.

Figures 4(a) and 4(b) display the variation of conductive temperature ϕ versus the radial distance r. From the figures, it is observed that at both the boundaries $r = 1$ and $r = 2$ the magnitude of the conductive temperature is 1, where the step-input temperature is imposed. We find that initially the temperature field shows the maximum value at both boundaries and it decreases with increase of radial distance towards the middle, becoming minimum at the middle except 1T3P model at $t = 0.35$. At lower time, the effect of diffusion is not significant of this field, but as time passes, a mild effect is observed under 3P, GN-III and LS models. However, GN-III and LS models predict very close values of the field at higher time but at lower time all three models (3P, GN-III, LS) – show significantly different values for both $\omega = 0.0$ and $\omega = 0.1$. This field also increases with the increase of time in all cases.

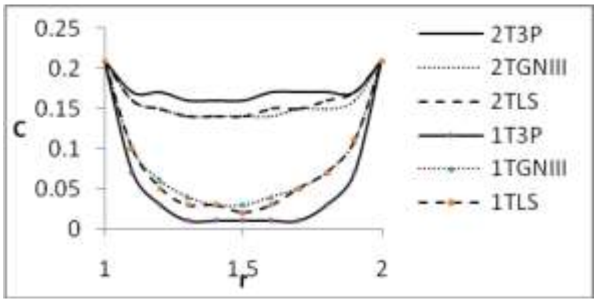

Fig. 5(a): Distribution of mass concentration at $t = 0.026$.

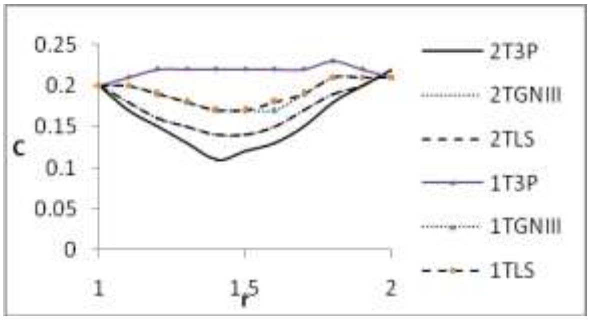

Fig. 5(b): Distribution of mass concentration at $t = 0.35$.

Figures 5(a) and 5(b) show the variation of mass concentration (C) against radial distance r inside the spherical shell for the thermoelastic diffusive medium. Like the temperature field, under all the three models (3P, GN-III, LS) the mass concentration shows maximum value near the boundaries and it decreases with increase of radial distance towards the middle and becomes minimum at the middle of the shell except 1T3P model at $t = 0.35$.

The 3P model predicts a significantly different value as compared to the other two models for both types of temperature field. As time passes, the difference between GN-III and LS models for this field decreases and they vary in exactly the same way. The magnitude of C is large for $\omega = 0.1$ at $t = 0.026$ and for $\omega = 0.0$ at $t = 0.35$ for all three models (3P, GN-III, LS). The values of this field increase with the time.

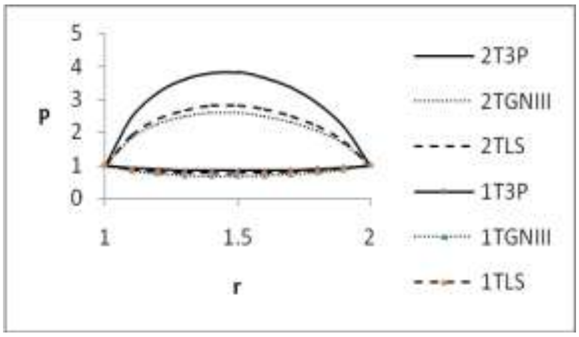

Fig. 6(a): Distribution of chemical potential at $t = 0.026$.

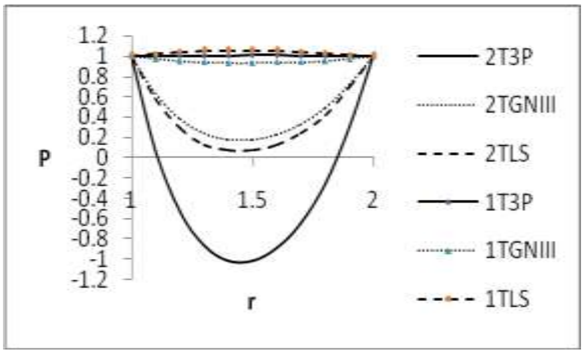

Fig. 6(b): Distribution of chemical potential at $t = 0.35$.

Figures 6(a) and 6(b) are plotted to study the variation of chemical potential P versus r for both small and large time in the case of three different thermoelasticity models (3P, GN-III, LS) by taking $\omega = 0.0$ and $\omega = 0.1$. The chemical potential is noted to be 1 at both the boundaries, which agrees with the initial boundary conditions. At both lower and higher time, the difference between chemical potential are more prominent under 2T3P, 2TGN-III, 2TLS models as compared to 1T3P, 1TGN-III, 1TLS models.

In all the figures, the result agrees with that of [18] and [42] for LS model considering thermoelastic diffusive medium.

IV. CONCLUSIONS

The problem of investigatingthe radial stress, shear stress, displacement, conductive temperature, mass concentration and chemical potentialin an isotropic elastic homogeneous spherical shell of the thermoelastic diffusive medium is studied in the lightof 2TLS, 2TGN-III and 2T3P models. We also compare our results with the corresponding results of one-temperature. The analysis of the results permit some concluding remarks:

1. The presence of diffusion plays an important role in all the physical quantities. It is observed that the

influence of diffusion is more significant on radial stress, shear stress and displacement, as compared to temperature field, for our present problem. We also noticed that the influence of diffusion on σ_{rr} and $\sigma_{\theta\theta}$ is more significant near the boundaries at lower time, whereas with increase of time the region of influence shifts towards the middle of the shell.
2. The significant differences in the physical quantities are observed for all the one-temperature models and two-temperature models. Two-temperature theory is more realistic than the one-temperature theory in the case of generalized thermoelasticity.
3. This paper is established based on 3P model. The assumption of 3P model that includes three-phase-lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. This is more general model that reduces to the GN-III and LS models as special cases.
4. The results analyzed through this problem should be beneficial for researchers working in the fields of material science, low temperature physics, design of new materials, geophysics and other industrial applications.

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